Wavelets and Autocorrelations

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Autocorrelations and Power Spectra

autocorrelation distributions:

$$\tilde{A}_n = 1/M \sum_{i=1}^{M} m_i \cdot m_{i+n}^* \quad \text{periodic system: } m_i = m_{i+\tau} \quad \phi$$

$$A_n = 1/(M-n) \sum_{i=1}^{M-n} m_i \cdot m_{i+n}^* \quad \text{aperiodic system} \quad \eta$$

discrete Fourier transform DFT:

$$\hat{m}_k(\delta x) = 1/\sqrt{M} \sum_{i=1}^{M} m_i(\delta x) e^{-\mathbf{i}\frac{2\pi}{M}k \cdot i}$$

DFT power spectrum:

$$\hat{P}_k(\delta x) = \hat{m}_k \cdot \hat{m}_k^*$$

product of averages over space and events

Wiener-Khinchine theorem, W-K T:

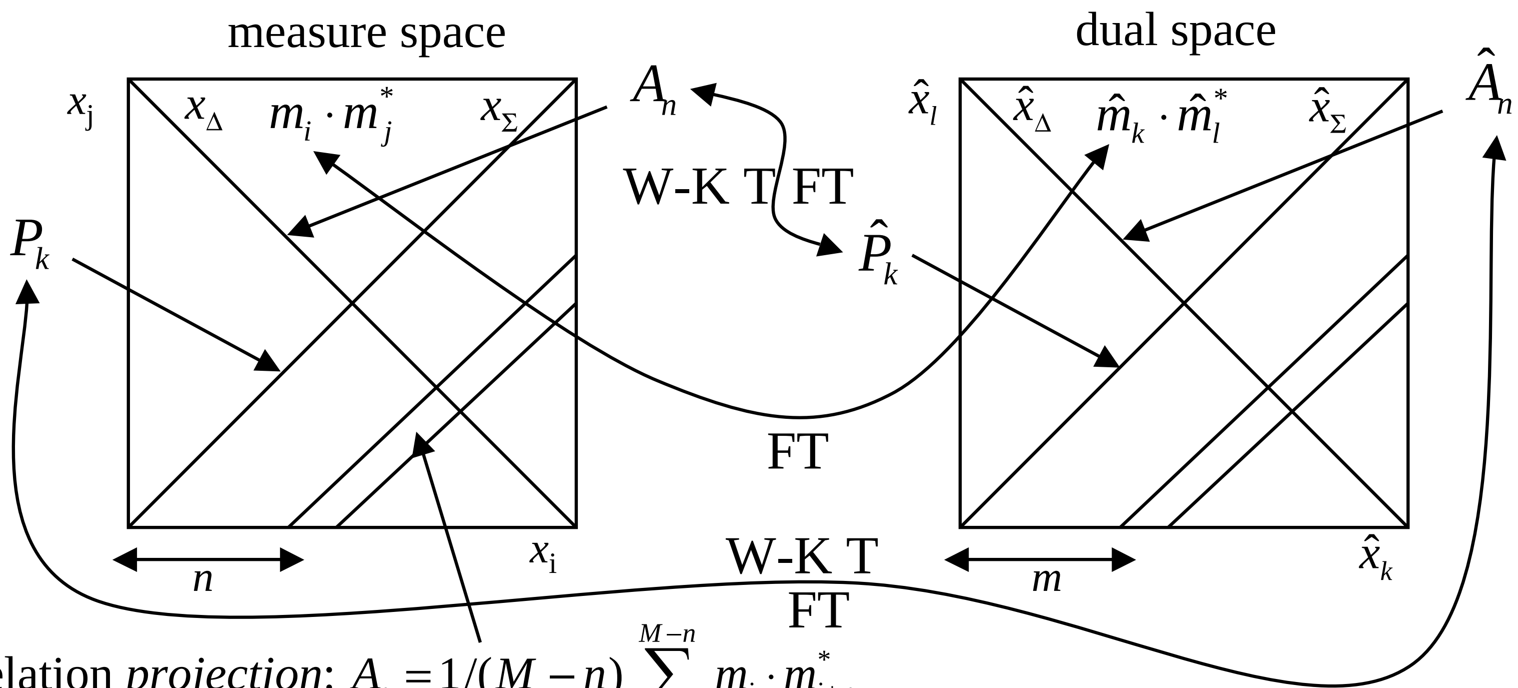
$$\hat{P}_k = \sum_{n=0}^{M-1} A_n e^{\mathbf{i}\frac{2\pi}{M}k \cdot n} \qquad A_n = \sum_{k=1}^{M} \hat{P}_k e^{-\mathbf{i}\frac{2\pi}{M}n \cdot k}$$

central element of time-series analysis

- Two forms of autocorrelation relevant to STAR
- Autocorrelations carry all relevant information

- Conventional form of power spectrum, from FT
- Wiener-Khinchine theorem couples power spectrum and autocorrelation

DFT Dual-Space Geometry



autocorrelation *projection*: $A_n = 1/(M-n) \sum_{i=1}^{M-n} m_i \cdot m_{i+n}^*$

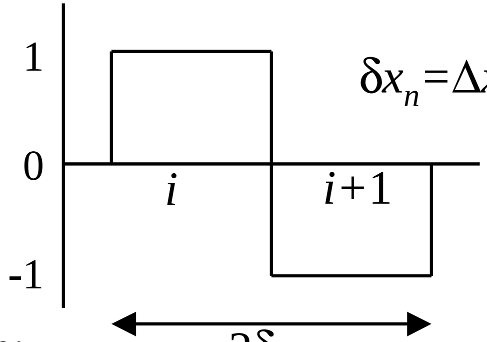
- The Fourier transform connects elements in a pair of dual spaces
- Each autocorrelation lies on a difference variable
- Each power spectrum lies on a sum diagonal
- The Wiener-Khinchine theorem connects two diagonals in two ways

Discrete Wavelet Transform

1D Haar wavelet:

$F_i(2\delta x)$

$M/2$ elements



$\delta x_n = \Delta x / M_n$

DWT 'discrete' transform:

bin number $M_n = 2^n$; n=1,…

'fineness' is index n

scale = δx_n

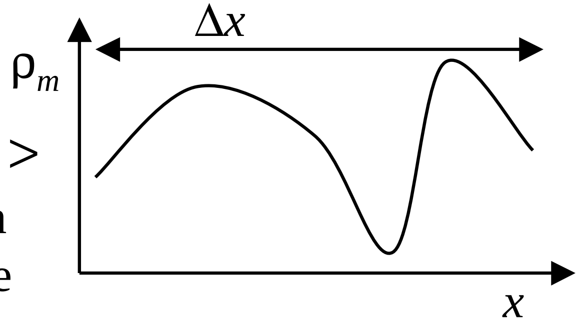
'Power spectrum:'

$$P(2\delta x) = 2/M \sum_{i=1}^{M/2} < \rho_m, F_i(2\delta x) >^2$$

$$= 2/M \sum_{j=1,odd}^{M-1} \{m_j(\delta x) - m_{j+1}(\delta x)\}^2$$

average of products over space and events

measure distribution:



$< \rho_m, F_i(2\delta x) >$

father function (FF) amplitude

- Consider Haar wavelets and measure density ρ_m on binned 1D space x
- WT is wavelet transform - *invertible* transform

- Transform is set of $<\rho, F_i>$
- DWT 'power spectrum' is expressed explicitly above in terms of M bin contents m_j at scale δx

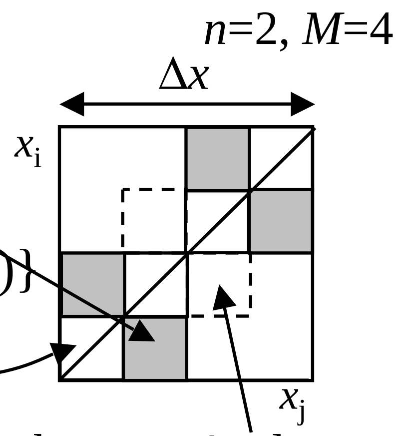
Wavelets and Autocorrelations

DWT ‘power spectrum:’

$$P(2\delta x) = 2/M \sum_{j=1,odd}^{M-1} \{m_j(\delta x) - m_{j+1}(\delta x)\}^2$$

$$= 2/M \sum_{j=1,odd}^{M-1} \{m_j^2(\delta x) + m_{j+1}^2(\delta x) - 2\, m_j(\delta x) \cdot m_{j+1}(\delta x)\}$$

$$\approx 2\{A_0(\delta x) - A_1(\delta x)\}$$



P misses the even A_1 elements

connection between DWT ‘power spectrum’ and autocorrelation

net autocorrelations

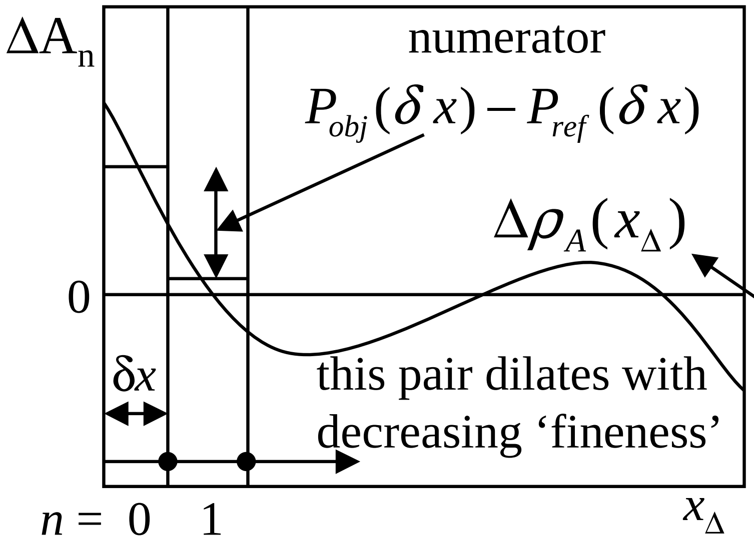
$$\{P_{obj}(2\delta x) - P_{ref}(2\delta x)\}/P_{ref}(2\delta x)\} \approx$$

$$\{\Delta A_0(\delta x) - \Delta A_1(\delta x)\} \;/\; \{A_{0,ref}(\delta x) - A_{1,ref}(\delta x)\}$$

- Expand DWT ‘power spectrum,’ obtain difference of two elements of the autocorrelation
- Differential WT measure: ratio of *net* autocorrelation scale differences to autocorrelation differences

Wavelet Geometrical Interpretation

net autocorrelation binned at δx



net autocorrelation density

$$\Delta A_n(\delta x) = \int_{n\delta x}^{(n+1)\delta x} \Delta\rho_A(x_\Delta)\, dx_\Delta$$ bin contents

$$\Delta A_0(\delta x) = \Sigma^2(\delta x)$$ total variance

$$(P_{obj} - P_{ref})/2 = 2\Sigma^2(\delta x) - \Sigma^2(2\delta x)$$

if total variance is homogeneous in δx ‘power spectrum’ is zero

understanding the DWT ‘power spectrum’ differential measure

- The numerator is a finite difference applied to bin integrals of the net-autocorrelation density - a 2x scale change
- Same applies to the reference autocorrelation in the denominator - over some scale interval it is approximated by variance

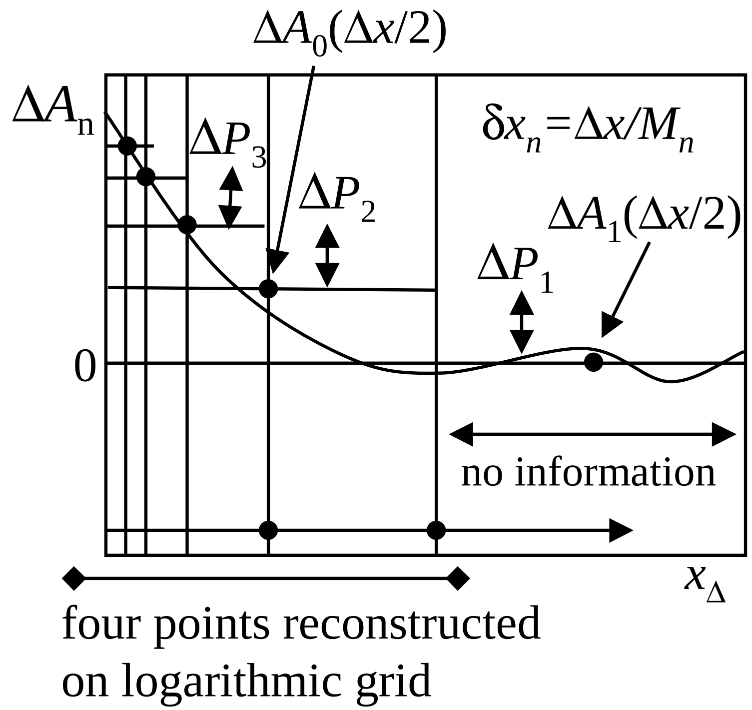
Reconstructing the Autocorrelation

Differential measure *numerator*:

$$\Delta P_n(2\delta x_n) \equiv P_{obj} - P_{ref}$$
$$\approx \Delta A_0(\delta x_n) - \Delta A_1(\delta x_n)$$

from this numerator we can reconstruct the net autocorrelation at successively smaller scales, in *octave* steps only

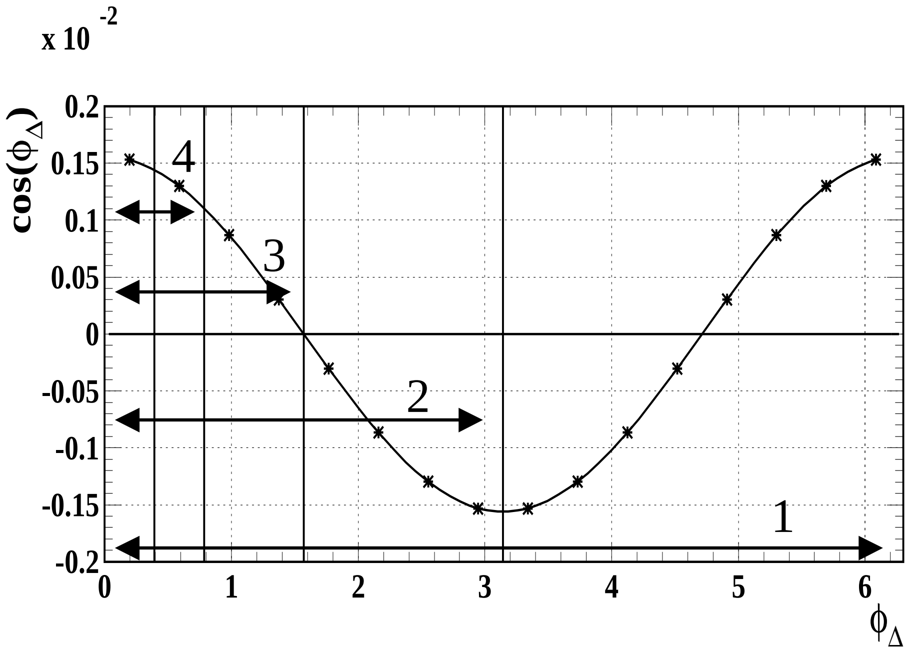
on the other hand, a scaling analysis of variance difference gives the same quantity at arbitrary scale steps



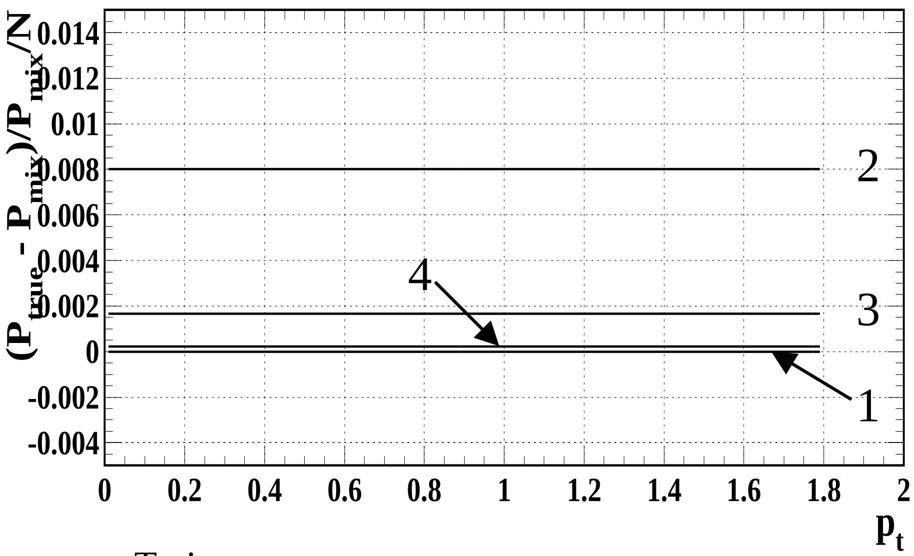
- The numerator of the differential measure can be used to reconstruct the autocorrelation as shown
- Denominator doesn’t help

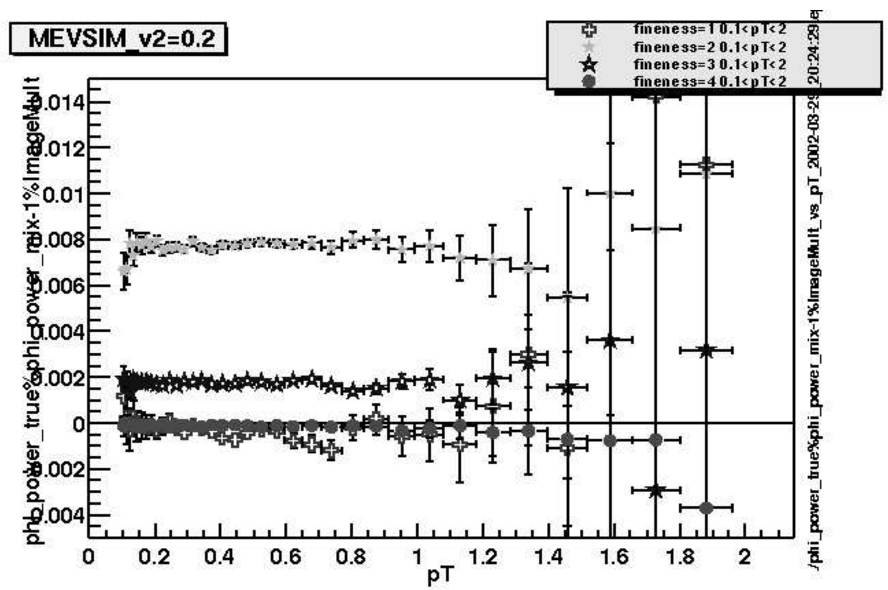
- Information in HI collisions is typically not on a logarithmic grid
- One learns more and more about less and less

Example



- Elliptic flow simulation
- Direct extraction of ‘power spectrum’ elements from cosine
- Matches wavelet analysis





DWT/DFT Accounting

- For a 'multiresolution' analysis on 16 bins with 4 scale steps the number of FF amplitudes is 15 (+1)
- The number of independent DFT amplitudes is also 16
- Both are invertible
- The number of aperiodic autocorrelation elements is also 16, on a linear grid
- The DWT 'power spectrum' contains four elements - on a logarithmic grid
- From this a *subset* of four autocorrelation points can be reconstructed on the grid
- Information contained in the DWT 'power spectrum' is generally much less than in the DFT autocorrelation

Conclusions

- The discrete wavelet transform (DWT) is a very successful lossless data compression scheme
- In HI applications sums of squared FF (father function) amplitudes define a ‘power spectrum’
- This is not a standard power spectrum
- The DWT ‘power spectrum’ is actually finite differences of the binned autocorrelation at an octave grid of scale points
- The autocorrelation itself or its FT the power spectrum is the optimal representation of two-point correlation structure

The DFT power spectrum is the product of averages
The DWT ‘power spectrum’ is an average of products